Discrete Mathematics - CSCE 531 Fall 2018   
In-Class Work, Day 01 (1 October 2018)

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet two inhabitants: and .

Notes:

* Each of the following exercises is a different scenario.
* The solutions will be given in terms of the propositions representing the statement “Inhabitant is a knight,” and representing the statement “Inhabitant is a knight.”

1. tells you that is a knave. says, “Neither nor I is a knave.” Determine who is a knight and who is a knave.

Most student solutions to this problem are probably far shorter than this one. In fact, many students could solve this problem easily in their heads and simply write down the solution. There are good reasons to think carefully about the steps involved, though. First, the same methods are applicable to qualitatively similar problems that are too complicated to solve in your head. Second, if one seeks to automate reasoning processes (e.g. for artificial intelligence or theorem proving), one must understand the steps well enough to describe them algorithmically. With that said …

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| Original Statement | Expansion Based on the Rules of the Island | Propositional Logic Representation |
| “ tells you that  is a knave.” | There are two cases to consider: either is a knight or is not a knight. If is a knight, then tells the truth, so ’s statement that is a knave is true, so is not a knight. On the other hand, if is not a knight, then is a knave, and therefore always lies. In this case, ’s statement is false, so it is not the case that is a knave, i.e. is a knight. |  |
| “ says ‘Neither  nor I is a knave.’” | [Similar reasoning, but slightly more succinct presentation.] If is a knight, then neither nor is a knave, i.e. both and are knights. If is a knave, then it is not the case that neither nor is a knave, i.e. either or is a knave (or both). |  |

We must determine “who is a knight and who is a knave” so that the “original statements” in the table above are (simultaneously) possible according to the rules of the island. Equivalently, we must find truth assignments for the propositions and so that the “propositional logic representations” above are both true. In other words, we need truth assignments such that (1) both  and  are true, and simultaneously (2) both and are true. Some simplification is possible, but in the interest of expediency, we can find our solution directly through the use of the following truth table:

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The only case for which the required propositions all hold is the one in which is true and is false. Therefore, is a knight and is a knave.

1. tells you that “of and I, exactly one is a knight.” tells you that only a knave would say that is a knave. Can you determine who is a knight and who is a knave?

We are given that “ tells you that ‘of and I, exactly one is a knight.’” ’s statement is equivalent to “Either (1) is a knight and is not a knight, or (2) is not a knight and is a knight.” Note that the two possibilities are mutually exclusive, so the exclusive “either-or” may be safely represented as the disjunction . Now, if is a knight, then ’s statement must be true; if is not a knight, then ’s statement must be false. Thus, it must be the case that

which is the same as

This can be simplified through the application of logical equivalences. It is typical to show fewer intermediate steps than I do here.

We begin by using the rule that is logically equivalent to for all propositions and :

Using the associativity of disjunction to remove the “extra” grouping symbols as well as the double negation yields

Using the associativity of disjunction and the absorption of disjunction over conjunction, we obtain

Next, we distribute the disjunctions over the conjunctions:

Sensing the end is near, we apply the negation law for disjunctions, leaving

Finally, the identity law for conjunction gives us

We are also given that “ tells you that only a knave would say that is a knave.” This is equivalent to “ tells you that if and only if is a knave, would say that is a knave. The translation of the last phrase (“ would say that is a knave”) takes the now familiar form , which we will temporarily abbreviate with the proposition . Translating the penultimate phrase (“if and only if is a knave”) takes a close variation of that form: , and we will abbreviate this compound proposition with . Finally, translating the first phrase is now routine: . However, using the fact that there is no actual Inhabitant , we can eliminate from

Similarly, we can eliminate first and then from :

Substituting for in the translation of the original statement yields:

Thus, we seek truth assignments for and such that both and are true, i.e. all of , , , and are true.

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There is exactly one assignment of truth values to and that satisfies the required compound proposition: and must both be false. Thus, inhabitants and must both be knaves.

1. A says that B is a knave. B says, “A and I are knights.” Can you determine who is a knight and who is a knave?

“A says that B is a knave.”

(A -> ~B) ˄ (~A -> B)

(~A ˅ ~B) ˄ (A ˅ B)

“A and I are knights.”

(B -> A ˄ B) ˄ [~B -> ~(A ˄ B)]

(B -> A ˄ B) ˄ [~B -> (~A ˅ ~B)]

[~B ˅ (A ˄ B)] ˄ [B ˅ (~A ˅ ~B)]

[(~B ˅ A) ˄ (~B ˅ B)] ˄ [B ˅ (~B ˅ ~A)]

[(~B ˅ A) ˄ (~B ˅ B)] ˄ [(B ˅ ~B) ˅ ~A]

[(~B ˅ A) ˄ T] ˄ [T ˅ ~A]

[~B ˅ A] ˄ [T]

(~B ˅ A)

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| A | B | ~A | ~B | ~A ˅ ~B | A ˅ B | ~B ˅ A |
| T | T | F | F | F | T | T |
| **T** | **F** | **F** | **T** | **T** | **T** | **T** |
| F | T | T | F | T | T | F |
| F | F | T | T | T | F | T |

There is exactly one assignment of truth values to A and B that satisfies the required compound proposition: A must be true and B must be false. Thus, we are able to determine who is a knight and who is a knave: Inhabitant A must be a knight and B must be a knave.

1. A claims, “B and I are not the same.” B says, “Of A and I, exactly one is a knight.” Can you determine who is a knight and who is a knave?

“A claims, ` B and I are not the same.'

First, consider the statement “A and B are not the same.”

(A -> ~B) ˄ (~A -> B)

(~A ˅ ~B) ˄ (A ˅ B)

Now, consider that the previous statement is a claim made by Inhabitant A.

{A -> [(~A ˅ ~B) ˄ (A ˅ B)]} ˄ {~A -> ~[(~A ˅ ~B) ˄ (A ˅ B)]}

{~A ˅ [(~A ˅ ~B) ˄ (A ˅ B)]} ˄ {A ˅ ~[(~A ˅ ~B) ˄ (A ˅ B)]}

{[~A ˅ (~A ˅ ~B)] ˄ [~A ˅ (A ˅ B)]} ˄ {A ˅ [~(~A ˅ ~B) ˅ ~(A ˅ B)]}

{[(~A ˅ ~B)] ˄ [T]} ˄ {A ˅ [(A ˄ B) ˅ (~A ˄ ~B)]}

{~A ˅ ~B} ˄ {[A ˅ (A ˄ B)] ˅ [A ˅ (~A ˄ ~B)]}

(~A ˅ ~B) ˄ {A ˅ (A ˄ B) ˅ A ˅ (~A ˄ ~B)}

(~A ˅ ~B) ˄ {A ˅ (~A ˄ ~B)}

(~A ˅ ~B) ˄ {( A ˅ ~A) ˄ (A ˅ ~B)}

(~A ˅ ~B) ˄ (A ˅ ~B)

(~A ˄ A) ˅ ~B

~B

“B says, `Of A and I, exactly one is a knight.'”

First, consider the statement “Of A and B, exactly one is a knight.”

(A -> ~B) ˄ (~A -> B)

This is identical to A’s claim. Following the same line of reasoning as above, the fact that the statement is a claim made by Inhabitant B therefore reduces to the requirement that ~A is true. Combining these results, we see that both Inhabitant A and Inhabitant B must be knaves, and we can answer the “Can you determine” question in the affirmative.

1. A tells you, “At least one of the following is true: that I am a knight or that B is a knight.” B claims, “A would say that I am a knave.” Can you determine who is a knight and who is a knave?

A tells you, `At least one of the following is true: that I am a knight or that B is a knight.'

[A -> (A ˅ B)] ˄ [~A -> ~(A ˅ B)]

[~A ˅ (A ˅ B)] ˄ [A ˅ (~A ˄ ~B)]

[T] ˄ [A ˅ ~B]

A ˅ ~B

A would say that B is a knave.

(A -> ~B) ˄ (~A -> B)

This is identical to the inhabitants’ claims in the previous problem. It is a claim made by B, and therefore reduces to the requirement that ~A is true. Conjoining this with A ˅ ~B yields ~A ˄ ~B, so both A and B must be knaves.

1. A says, “B and I are both knights or both knaves.” B claims, “A and I are the same.” Can you determine who is a knight and who is a knave?

B and A are both knights or both knaves.

(A ˄ B) ˅ (~A ˄ ~B)

A says, `B and I are both knights or both knaves.'

{A -> [(A ˄ B) ˅ (~A ˄ ~B)]} ˄ {~A -> ~[(A ˄ B) ˅ (~A ˄ ~B)]}

{~A ˅ (A ˄ B) ˅ (~A ˄ ~B)} ˄ {A ˅ [~ (A ˄ B) ˄ ~(~A ˄ ~B)]}

(~A ˅ B) ˄ {A ˅ [(~A ˅ ~B) ˄ (A ˅ B)]}

(~A ˅ B) ˄ {[A ˅ (~A ˅ ~B)] ˄ [A ˅ (A ˅ B)]}

(~A ˅ B) ˄ (A ˅ B)

B

Thus, B must be a knight.

A and B are the same.

(A -> B) ˄ (~A -> ~B)

(~A ˅ B) ˄ (A ˅ ~B)

[(~A ˅ B) ˄ A] ˅ [(~A ˅ B) ˄ ~B]

[(~A ˄ A ) ˅ (B ˄ A)] ˅ [(~A ˄ ~B) ˅ (B ˄ ~B)]

(B ˄ A) ˅ (~A ˄ ~B)

This is identical to the statement that B and A are both knights or both knaves. The fact that B makes this claim implies that A is a knight. We have successfully determined who is a knight and who is a knave.